Characterization of polymeric structural foams under compressive impact loading by means of energy-absorption diagram

M. Avalle, G. Belingardi, R. Montanini

Abstract

The mechanical properties at room temperature of three polymeric foams (namely EPP, PUR and PS/PA foams) have been experimentally evaluated in both static and impact loading conditions. The energy absorption characteristics have been examined both through the energy-absorption diagram method and through the efficiency diagram method. The meaning of the efficiency parameter, already used in the literature, has been explained in a proper, satisfactory way. It is shown that the maximum of the efficiency identifies the condition for optimal energy absorption of the foam, while the maximum stress reaches a value limited through other design considerations. The efficiency diagram method is then used to obtain synthetic diagrams useful to characterize the material and to help the design of energy absorbing components. These synthetic selection diagrams are obtained for the three tested materials. Finally, some consideration are drawn comparing the mechanical performance of the three considered types of foams and their dependency on density. © 2001 Elsevier Science Ltd. All rights reserved.

1. Introduction

Energy-absorbing polymeric foams are widely used in the automotive industry to prevent injuries to the occupants in the event of front or side collisions. The use of foamed materials result in a significant improvement in the passive safety of the vehicle, owing to their excellent energy dissipation properties. In addition, they have low apparent density and are relatively cheap, and
allow great design flexibility, as they can be easily modeled in complex geometric parts. Typical applications include seat cushions, bumper systems, instrument panels and side impact protection systems. Effort is currently underway to minimize occupant head injury during an impact. The severity of injuries can be prevented or at least mitigated by the use of structural foams as cushioning.

An optimum energy-absorbing material needs to dissipate the kinetic energy of the impact while keeping the force on it below some limit, thus, resulting in a no-dangerous deceleration on the occupants. The design of effective cushioning involves many other considerations, like the shape of the protective structure, which influences load transfer during impact, and the capacity to absorb elastic energy, which controls rebound.

Polymeric foams are well suitable for this kind of applications. They can undergo large compressive deformations and absorb considerable amounts of specific energy. Energy is dissipated through the cell bending, buckling or fracture, but the stress is generally limited by the long and flat plateau of the stress–strain curve. This behavior explains the high-energy efficiency that can be obtained with foamed materials. Moreover, for the same amount of dissipated energy, the foam specimen always gives a maximum force lower than the corresponding solid specimen of equal volume made of the material from which the foam is derived.

Fig. 1 shows a typical nominal (based on the initial dimensions of the sample) stress–strain curve for a rigid polyurethane foam, obtained by compressing a cubic specimen quasi-statically along one direction. The curve exhibits three definite regions: linear elasticity, plateau and densification. At small strains, usually less than 5%, the behavior is linear elastic, with a slope equal to the Young modulus of the foam. As the load increases, the foam cells begin to collapse by elastic buckling, plastic yielding or brittle crushing, depending on the mechanical properties of the cell walls. Collapse progresses at roughly constant load, giving a stress plateau, until the opposing walls in the cells meet and touch, when densification causes the stress to increase steeply. In the unloading
phase, the stress varies non-linearly with the strain. The work done per unit volume (specific energy) in deforming the foam to a given strain $\varepsilon$ is simply the area under the stress–strain curve up
to the strain $\varepsilon$. Very little energy is absorbed in the linear elastic region; it is the long plateau of the stress–strain curve that allows large energy absorption at near constant load.

Several studies have been undertaken to experimentally characterize the impact properties of polymeric foams for potential automotive application. Data have been reported for foams obtained from polyurethane [1–17], polystyrene [3,4,5,11,12,18], polypropylene [3,11–16,21–24], polyethylene [4,5,8], ABS [4,5,22], phenolic [4,5] and olefinic [19] resins, but results are quite scattered and not sufficiently general. As a consequence, the practical use of energy-absorbing foams is still characterized by a high degree of empiricism.

2. Experimental program

2.1. Materials

Three different foam materials were considered for the test program:

(a) expanded polypropylene (EPP) supplied by MONTELL, in five different nominal densities (30, 45, 70, 106 and 145 kg/m$^3$);
(b) rigid polyurethane foam (PUR) supplied by BAYER, in two nominal densities (70 and 100 kg/m$^3$);
(c) a foam obtained from a blend of polyamide reinforced with modified polyphenylene and polystyrene (NORYL GTX®) supplied by GE Plastics, in two nominal densities (50 and 75 kg/m$^3$).

A microscopic inspection reveals that cell distribution and orientation are almost uniform for all the examined materials, thus suggesting that the mechanical behavior is independent from the direction of loading. In Fig. 2 microphotographs of the three materials obtained by SEM at same magnification are reported. EPP shows a structure of closed cells. PUR foam also has closed cells but the dimensions of the cells are remarkably smaller than those of EPP foam. On the contrary, NORYL GTX® foam is based on an open cell structure. This is demonstrated by the picture in Fig. 2d at a much higher magnification (one order of magnitude).

2.2. Specimens

For each material, six specimens, having nominal dimensions of $50 \times 50 \times 50$ mm (Fig. 3), were cut from a foam plate. The size of the specimen was chosen to obtain a reasonable compromise between the maximum amount of relative compression of the entire group of specimens. Since the machine used for the dynamic tests has a fixed amount of energy available for a given impact speed, lower density foam specimens experienced much greater compression than higher density foam specimens. Besides, it was verified that moderate variations of specimen size do not affect the results. For each density, quasi-static and dynamic compression tests were conducted to measure the mechanical response of the foams under impact. Each test was repeated three times under the same nominal condition (i.e. same compression rate and temperature) to determine the significance
Fig. 2. Microphotographs of the three materials tested (magnification 25×): (a) expanded polypropylene (EPP); (b) rigid polyurethane (PUR); (c) polystyrene/polyamide reinforced with modified polyphenylene (NORYL GTX®); (d) NORYL GTX magnified 250×.

Fig. 3. Sample cubic specimen (50 mm side) of the three materials: (a) expanded polypropylene (EPP); (b) rigid polyurethane (PUR); (c) polystyrene/polyamide reinforced with modified polyphenylene (NORYL GTX®).

of response variability. Test data showed that the repeatability of the three tests was excellent for the quasi-static loading and good for the dynamic loading.

The structural response of the foam is strongly influenced by the cell geometry (foam apparent density $\rho_f$, cell topology and anisotropy ratio) and by properties of the solid material
forming the cell walls (solid material density $\rho_s$, Young modulus $E_s$, yield strength $\sigma_{ys}$, ultimate strength $\sigma_{us}$).

Specimens were accurately measured and weighted before testing, and the average apparent density was calculated for each foam. Nominal properties of the solid material could not be measured and were supplied by the manufacturers. Table 1 summarizes the main physical and mechanical properties of the analyzed foams.

### Table 1

Physical and mechanical properties of the examined foams

<table>
<thead>
<tr>
<th>Material</th>
<th>$\rho_l$ nominal (kg/m$^3$)</th>
<th>$\rho_l$ measured (kg/m$^3$)</th>
<th>$\rho_s$ (kg/m$^3$)</th>
<th>$E_s$ at 20°C (MPa)</th>
<th>$\sigma_{ys}$ (MPa)</th>
<th>$\sigma_{us}$ (MPa)</th>
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<tr>
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<td>950</td>
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<td>950</td>
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<td>73.5</td>
<td>1150</td>
<td>1600</td>
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2.3. Testing methods

Foam specimens were tested under quasi-static and dynamic compression loading, to determine their energy absorption characteristics and impact behavior.

Quasi-static tests were conducted on an electronically controlled universal testing machine (DARTEC 9600). Foams were compressed at constant velocity between two steel plates. One plate was supported by a spherical joint. Force, time and piston displacement were recorded during the test and used for the data elaboration. The maximum stroke was chosen to obtain a maximum specimen compression of about 90% the initial thickness. The actuator speed was fixed at 60 mm/min, giving a specimen strain-rate of 0.02 s$^{-1}$. The quasi-static test procedure is consistent with ASTM D1621/94 for rigid cellular plastics [25].

Polymeric foams often exhibit very high strain-rate dependence compared to solid metallic materials. This dependence is due to the solid material properties and to the presence of a fluid, generally air, inside the foam. When the foam deforms, the air is compressed or forced outside, depending on the foam cellular structure (closed cells or open cells, respectively). The fluid flow is strongly influenced by the compression rate, therefore, especially for passive safety-related applications, an accurate knowledge of the dynamic behavior is necessary for an effective design. While quasi-static tests are useful to understand the influence of some design parameters like the relative density, this information should be completed with a quantification of foam strain-rate sensitivity that can be evaluated only by proper dynamic tests.

Strain-rates up to 100 s$^{-1}$ are quite common in impact applications. Such values cannot be obtained with a universal testing machine and a special apparatus is needed. In this work, dynamic
tests on foams were performed with a drop dart machine (Fall-O-Scope). The machine has a mass of 20 kg and a maximum drop height of 2 m. The drop dart test does not allow a constant velocity of the impacting mass during the whole compression stroke, thus quasi-static and dynamic results may not be directly compared. However, the speed reduction is important in the densification phase where force and deceleration are continuously growing. For example, in a typical expanded polypropylene foam sample test the speed reduction is only 10% at a relative compression of 60%. Hence, results for the plateau phase, which is the most interesting part of the mechanical characteristic from the point of view of energy absorption, can be considered only slightly affected by speed reduction. In the dynamic tests, the force was measured by a piezoelectric load cell, 100 kN maximum load, while the velocity and the displacement were obtained by integrating the acceleration with respect to time. Data were acquired through an analog-to-digital converter card installed on a personal computer. No filtering of the data was applied. By simple variation of the initial drop height, it is possible to change the strain-rate that the specimen is subjected to. However, in this way the final displacement is also changed, because the potential energy of the impacting mass is changed. Another way to change the applied strain-rate while maintaining the same final displacement is to modify both the foam thickness and the drop height to maintain constant the final compression of the foam. This method has not been used here. However, it has been verified that the results are not significantly influenced.

2.4. Interpretation and utilization of the results from testing of foams

As previously mentioned, the aim of packaging or shock absorbing is to dissipate the kinetic energy of the impacting mass while keeping the maximum force (or acceleration) below some limit. For each application, a foam with optimum value of the density can be found. If the density is too low, the densification zone is reached and a very high force is obtained before all the energy has been dissipated, while, if the density is too high, the force exceeds the critical value before enough energy has been absorbed, while the material compressive strain remain only partially utilized [20]. This idea can be understood by observing the graph of Fig. 4. In this figure, the stress–strain curves of three foams are compared. The hashed area under each curve corresponds to the same amount of the energy absorbed, the maximum strain and stress levels are consequently determined.

![Fig. 4. Energy diagram of typical foams of different density obtained from the same solid material.](image_url)
The lighter foam (with lower density $\rho_1$) is able to absorb the prescribed amount of energy with large deformations because there is a low-value plateau and the foam comes up to densification. On the contrary, the heavier foam (with higher density $\rho_3$) does absorb the same amount of energy with low deformation and high stresses. The ideal foam is that with intermediate density $\rho_2$. By plotting (as in right side of Fig. 4) the maximum stresses that are reached by the three foams to absorb the same prescribed amount of energy it is possible to determine the optimal density for the application.

To better apply the idea herein explained, it is useful to use the so-called energy-absorption diagram [20]. An energy-absorption diagram is obtained by plotting the absorbed energy $W$ as a function of the stress. The absorbed energy (which is the area under the stress–strain curve) is simply

$$W = \int_0^e \sigma(e) \, de. \quad (1)$$

By plotting together the absorbed energy-stress curves for the various foams it is easy to select the most adequate foam for the application (Fig. 5). Given a prescribed level of energy to be absorbed (represented by the horizontal line on the energy-absorption diagram of Fig. 5) the
Fig. 6. Efficiency of typical foams of different density obtained from the same solid material as in Fig. 4.

optimal density is the one that gives the minimum stress (stress values are the intersections of this horizontal line with the absorbed energy-stress curves, represented in the lower part of Fig. 5 as a function of the density).

The envelope of the absorbed energy-stress curves is a curve that represents points of optimum energy absorption whatever the value of the density.

Another way to present the data for the energy absorption of foams and to obtain indication for optimum usage is to plot the so-called efficiency parameter [8] as a function of the stress. The efficiency parameter is defined as the ratio of the absorbed energy up to the stress divided by the stress itself as follows:

$$E = \frac{\int_0^\sigma e \sigma(e) \, de}{\sigma}.$$  \hspace{1cm} (2)

Plotting the efficiency for each material as a function of the stress, it is possible to obtain a series of curves like those of Fig. 6. The efficiency has always a maximum at a certain stress because, beyond a certain stress level, the increase of absorbed energy is lower than the corresponding stress increase (this is easy to understand by observing the energy-absorption diagram of Fig. 5). For the efficiency curves too, it is possible to define an envelope. It is possible to demonstrate that this envelope is equivalent to the envelope of the energy-absorption diagram, and that both diagrams lead to the same results.

For a prescribed energy level let us define on the energy-absorption diagram the optimal density (say for example $\rho_2$). For that value of the absorbed energy, the maximum stress $\sigma_2$ reached in $\rho_2$ density foam is always less than the maximum stress in whatever foam of different density (see Fig. 5c). Consequently, the absorbed energy-stress curve of $\rho_2$ density foam is tangent to the envelope in $(\sigma_2, W_2)$ point ($W_2$ being the energy absorbed by $\rho_2$ density foam up to stress $\sigma_2$). On the other hand, if stress level $\sigma_2$ is fixed the energy absorbed by whatever foam of density different from $\rho_2$ is lower than the energy absorbed by $\rho_2$ density foam (see Fig. 5b). This is represented by the vertical line passing through the point $(\sigma_2, W_2)$ in Fig. 5a and by the diagram of Fig. 5b in which the energy absorbed up to stress $\sigma_2$ is plotted against density.

If we divide the $W$ value of this last curve by $\sigma_2$ we obtain efficiency $E$ for the same stress level $\sigma_2$ as a function of the density (as is shown in the diagram of Fig. 6b). As a consequence, if the
envelope curve is the locus of optimum points that maximize the absorbed energy for each value of the density, this must hold also for the efficiency envelope (Fig. 6a) because the curves correspond each other through the division by stress $\sigma$.

Hence, the efficiency diagram is equivalent to the energy-absorption diagram. However, the efficiency diagram has some advantages. First, it seems to be more “intuitive” in the sense that the foam density is chosen to maximize some sort of efficiency. The most interesting aspect is, however, that the efficiency diagram is easier to manage. This holds in particular, for the construction of the envelope curve. To obtain mathematically the envelope is not a very simple task. However, the envelope of the efficiency curve can be very well approximated by simply connecting the points of maximum efficiency for the different densities. In particular, it can be shown (the demonstration is given in the appendix) that when the envelope of the energy-absorption diagram is a straight line the tangency points correspond to point of maximum efficiency. Moreover, in this case the efficiency envelope is a constant.

The previous discussion has a sound basis if the response of the material is characterized by the stress monotonically increasing with strain. If the material characteristic shows a relative maximum in an early stage of deformation the use of the efficiency is questionable [26,27], at least if the maximum is very high. In this case the total efficiency, ratio of energy $W$ over the maximum experienced stress, could be more suitable. This is not the case for most polymeric materials and in particular for the three types that are considered in this work. Finally, it should be noted that if the stress is monotonically increasing, the efficiency and the total efficiency are identical.

Another indicator of the foam usage effectiveness is the so-called ideality parameter [8] defined as the ratio of the absorbed energy $W$ to the product of the actual stress and strain:

$$I = \frac{\int_0^\varepsilon \sigma(e) \, de}{\sigma \varepsilon}.$$  

(3)

Hence, ideality $I$ is also the ratio of the efficiency to the maximum strain value. Ideality $I$ is apparently a good indicator of the efficient usage of the foam up to stress $\sigma$. In fact, ideality $I$ is the ratio of the energy that is absorbed up to stress $\sigma$ and strain $\varepsilon$ compared to the energy absorbed by an ideal absorber (which gives constant stress $\sigma$) up to the same strain $\varepsilon$. However, by analyzing the curves of ideality $I$, it is possible to observe that, despite the fact that there is a maximum ideality, the use of the foam energy-absorption capability obtained with this method is unsatisfactory (Fig. 7).

![Fig. 7. Ideality $I$ curves on the energy-absorption diagram of typical foams of different density obtained from the same solid material as in Fig. 4.](image)
In fact, maximum ideality corresponds to very low values of deformations, thus neglecting the energy absorption capability of the foam in the plateau branch of its mechanical characteristic. Maximum ideality roughly (not exactly) corresponds to the maximum slope of the absorbed energy-stress curves. Therefore, maximum ideality can be used to determine the lower limit for a proper usage of a foam.

Finally, the results obtained from the compression tests on different materials can be used to obtain synthetic diagrams to be directly used by the designer of an energy-absorbing device. Taking the values of stress for maximum efficiency obtained from different materials it is possible to obtain curves of the densities to be used for optimum energy absorption (see Fig. 8). In the same diagram, the corresponding absorbed energy (represented by another curve on a secondary ordinate axis) calculated for the point of optimum energy absorption is reported. By the way, this curve is exactly the envelope in the energy-absorption diagram. Therefore, the designer that needs to select a foam to work at a prescribed stress level, say $\sigma_2$, enters the diagram with that abscissa and finds on the primary ordinate axis the suitable density and on the secondary ordinate the specific energy absorbed by this foam at the optimum working stress. Sizing of the foam component will be determined from the amount of total energy to be dissipated.

As foams are extremely sensitive to temperature, strain-rate and other external factors the diagrams will be valid only in the cases of the corresponding well-determined testing condition.

Others methods have been suggested for characterizing energy absorption capability of foams, based on the Jansen factor [20], on the cushion factor [20] the Rush’s curve [5]. However, the methods analyzed here seem easy to use and have a sound conceptual basis therefore will be used throughout this work.

3. Results: expanded polypropylene (EPP)

In Fig. 9, experimental static ($\dot{\varepsilon} = 0.02 \text{s}^{-1}$) efficiency-stress curves are plotted for EPP foams having five different nominal densities (31, 45, 70, 106 and 145 kg/m$^3$). EPP efficiency is less than 40% and it decreases with foam density. It can also be noted that stress for maximum efficiency
increases non-linearly with foam density. In the figure, iso-energetic levels corresponding to maximum efficiency are also reported for each density.

Dynamic tests were performed on EPP foam samples at two different impact velocities that correspond to an initial strain-rate $\dot{\varepsilon}$ of, respectively, 2 and $60 \text{s}^{-1}$ (Fig. 10). Tests at $2 \text{s}^{-1}$ were run on the hydraulic testing machine, i.e., at constant speed. Tests at $60 \text{s}^{-1}$ were run on the drop-dart testing machine. Hence, this value of strain-rate is actually the initial value that can be calculated as...
the ratio of the impact velocity over the initial specimen length. This method for calculating the strain-rate has also been used for the tests on the other two materials. The effects of the speed reduction during the evolution of the test are quite limited before densification is achieved, as already discussed in a previous section. EPP foam characteristic was found to be quite sensible to the strain-rate: energy absorbed per unit volume at maximum efficiency is much higher when the foam is compressed dynamically, but the stress is higher too.

Density-stress and specific energy-stress diagrams at maximum efficiency are reported in Fig. 11. It can be deduced that, to fully exploit the material, foam density should be selected depending on the expected impact velocity. If the maximum efficiency criteria is adopted it derives that the strain-rate influence on the specific energy dissipated is much less pronounced.

4. Results: expanded polyurethane (PUR)

PUR samples with two different densities (70 and 100 kg/m$^3$) were tested under static and dynamic conditions. Three strain-rate levels were considered (0.2, 2 and 90 s$^{-1}$). PUR efficiency-stress curves are reported in Fig. 12 (static compression) and Fig. 13 (dynamic compression).

PUR foams exhibit a maximum efficiency of about 50% when compressed statically with small differences between the two different density values. Under dynamic loading, maximum efficiency is a little bit higher, although the energy absorbed per unit volume is not significantly affected by the impact velocity. PUR samples did not maintain the structural integrity because they cracked during compression. Thus results are more scattered if compared to the other types of foams.

Starting from the experimental data, a selection diagram was constructed as indicated in Fig. 14. Compared with EPP, PUR foams exhibit higher energy dissipation capability and are much less influenced by the strain-rate effects.
5. Results: polyamide reinforced (NORYL GTX®)

NORYL GTX® foams, Figs. 15 and 16, have good efficiency and can dissipate a high quantity of energy for unit volume at maximum efficiency. As it can be seen from the curves in Fig. 17, these foams are strain-rate dependent, but this dependence is less pronounced compared to EPP foams.
6. Conclusions

Cellular materials, generally known as structural foams, play an important role in many passive safety applications, from automotive to packaging. These materials have low cost, very low weight,
and high workability. Different types of polymeric, ceramic and even metallic foams, obtained from different types of base materials and with different degree of density, are available, each with peculiar characteristics, often not yet well known.

In this work, the mechanical properties at room temperature (mechanical characteristics of polymeric foams are generally extremely sensitive to temperature changes) of three polymeric foams (namely a polypropylene foam, a polyurethane foam and a polyamide foam) have been examined. Stress–strain curves have been obtained in both static and impact-loading conditions.
For polypropylene foam five different densities have been examined, whereas, only two densities have been considered for polyurethane- and polyamide-reinforced foams.

The energy absorption characteristics have been particularly examined both through the energy-absorption diagram method and through the efficiency diagram method. Starting from the measured compression stress–strain curve, for each foam, the efficiency curve can be computed. The meaning of the efficiency parameter, already used in paper [8], has been explained in a proper more satisfactory way. The maximum of the efficiency identifies the condition for optimal energy absorption of the foam, while the maximum stress reaches a properly limited value. It has been shown that the results obtained from the efficiency diagrams are consistent with the results obtained from the energy-absorption diagrams. However, efficiency diagrams seem to offer some practical advantage when the envelope line has to be drawn.

The efficiency diagram method has been used to obtain synthetic diagrams useful to characterize the material and to help the design of energy-absorbing components. This synthetic selection diagrams have been obtained for the currently tested materials.

Polypropylene and polyamide foams exhibit similar properties and are strongly sensitive to strain-rate although their cell structure is different. Polyurethane foams have a completely different behavior with a large intermediate plateau phase and very low sensitivity to the strain-rate. This is probably due to the different properties of the base material that cause failure through large plastic flow and fracture of the cell walls. It is worth noting that for these reasons an increase of the relative density seems not to have an important influence on their mechanical properties.

The wide plateau that has been observed in the stress–strain characteristic of PUR and polyamide foams justifies the higher efficiency values obtainable by these foams. However, while PUR foams lost their integrity during the compression, this is not the case of polyamide NORYL GTX® foams that are able to resist to multiple impacts.

Acknowledgements

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Appendix

First, it is necessary to write the maximum efficiency $E$ as a function of the stress $\sigma$:

$$E(\sigma) = \int_0^\sigma \frac{\sigma(e)(de/d\sigma)}{\sigma} d\sigma. \quad (A.1)$$

By deriving efficiency $E$ with respect to stress $\sigma$ the following expression is found:

$$\frac{dE}{d\sigma} = \frac{\sigma^2 de/d\sigma - \int_0^\sigma \sigma(e)(de/d\sigma) d\sigma}{\sigma^2}. \quad (A.2)$$
The maximum of the efficiency function with respect to \( \sigma \) is obtained equating to zero the numerator of the derivative. So the following equation is found:

\[
\frac{\sigma^2}{d\sigma} d\sigma = \int_0^{\sigma} \sigma(e) \frac{de}{d\sigma} d\sigma,
\]

(A.3)

\[
\frac{\sigma^2}{d\sigma} = W(\sigma),
\]

(A.4)

where \( W(\sigma) \) is the energy dissipated up to stress \( \sigma \).

A straight line passing through the origin of \( W-\sigma \) plane and tangent to the \( W(\sigma) \) curve is determined by the following conditions:

\[
k\sigma = \int_0^{\sigma} \sigma \frac{de}{d\sigma} d\sigma,
\]

\[
\frac{d}{d\sigma}(k\sigma) = \frac{d}{d\sigma} \left( \int_0^{\sigma} \frac{d\sigma}{d\sigma} d\sigma \right).
\]

(A.5)

Obviously, the second expression can be simplified as follows:

\[
\frac{d}{d\sigma}(k\sigma) = \frac{d}{d\sigma} \left( \int_0^{\sigma} \frac{d\sigma}{d\sigma} d\sigma \right) \rightarrow k = \frac{d\sigma}{d\sigma}.
\]

(A.6)

By substituting this expression in the first of (A.5), Eq. (A.4) is found.

Therefore, a straight line passing through the origin of \( W-\sigma \) plane is tangent to the \( W(\sigma) \) curve in the point that maximizes efficiency \( E \) with respect to stress \( \sigma \). As a consequence, if the envelope in \( W-\sigma \) plane is precisely a straight line, every point of this line will have a corresponding maximum efficiency point. Since it has been shown that a point of the envelope curve in the plane of the absorbed energy correspond to a point of the envelope curve in the efficiency plane, its value being equal to the absorbed energy divided by the stress, then the straight line in the plane of the absorbed energy correspond to a horizontal line in the efficiency plane.

References


